DebtRank and financial contagion

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Recap: Centrality measures

- Degree centrality:
  \[ x_i = k_i \]

- Eigenvector centrality:
  \[ x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j \]

- Katz centrality:
  \[ x_i = \alpha \sum_j A_{ij} x_j + \beta \]

- PageRank (Google, random surfer):
  \[ x_i = \alpha \sum_j A_{ij} \frac{x_j}{k_{jout}} + \beta \]
DebtRank: Too Central to Fail? Financial Networks, the FED and Systemic Risk

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The characterization of the architecture of economic and financial networks is gaining increasing importance\textsuperscript{1}. Indeed, the recent economic turmoil has raised a broad awareness that the financial system should be regarded as a complex network whose nodes are financial institutions and links are financial dependencies\textsuperscript{1}. In this perspective, systemic risk is meant here as the risk of a systemic default, i.e. the default of a large portion of the financial system. It can be quantified and measured from the analysis of the dynamical evolution of the nodes and from the structure of the network\textsuperscript{2-4}. The main open question regarding financial networks concerns the determination of the so-called “systemically important” financial institutions, namely, the ones that, if defaulting, can trigger a systemic default and are thus to be considered “too big to fail”. From a network science perspective, this question is related to the concept of recursive centrality measures such as eigenvector centrality and PageRank. It is also related to the more general issue of the controllability of a complex network\textsuperscript{5}. However, the investigation of how financial networks function and how systemic risk emerges is only at the beginning. The scarcity of data due to confidentiality constraints has limited so far the study to a few national datasets\textsuperscript{2-4}. The goal of this paper is to show how network science can contribute to a quantitative assessment of systemic risk. To this end, we analyse a unique and very relevant dataset by means of a novel indicator of systemic importance.

In the US, the financial crisis reached its peak in the period March 2008 – March 2010. During this time, many US and international financial institutions received aid from the US Federal Reserve Bank (FRD) through emergency loans programs, including the so-called “FED Discount Window”. The amount and the recipients of these loans were not disclosed until very recently (see more details in Supplementary Information, Section 1-2). This data represents, to our knowledge, the first data set publicly available, on the daily financial exposures between a central bank and a large set of institutions over several months. The data was previously used mainly from the point of view of accounting practice and conflicts of interests\textsuperscript{5}. Here, we instead present an analysis from the perspective of complex financial networks and systemic risk.

The contributions of this paper are the following. We first analyse the portfolio of loans granted by the FED overtime, both in terms of concentration and fragility. We then investigate the distribution of outstanding debt across institutions and across time. We also combine the FED dataset with data on equity investment relations among those institutions and we analyse the structure of the network of dependencies among the institutions that received funding. Finally, in order to estimate the systemic importance of the various institutions, we introduce Debt Rank. This is a novel measure, akin to feedback centrality, that takes into account in a recursive way the impact of the distress of one or more institutions to their counterparts across the whole network.

**Results**

Credit concentration and fragility. We start our analysis of the dataset (see Methods) from the sum of the outstanding debt across institutions, which represents the total exposure of the FED, i.e. its total potential loss in case of default of the borrowers. This number rose very sharply in November of 2008, up to around USD 1.2
Banks are linked via balance sheet identities

- External assets $A^E_i$, i.e. liquid (e.g. reserves) and illiquid investments (e.g. consumer loans) outside the financial sector
- Internal assets $A^I_i$ which are liabilities of other institutions

### Banking networks

\[
A^E_i + A^I_i = L^E_i + L^I_i + E_i
\]

\[
A^I_{ij} = L^I_{ji} \quad \text{i.e.} \quad \sum_i A^I_i = \sum_i L^I_i
\]
Banking network

Credit exposure:
- Amount $A_{ij}^l$ \textbf{invested} by $i$ into $j$
  - Total value of interbank assets: $A_i^l = \sum_l A_{il}^l$
  - Loss $A_{ij}^l$ when counterparty $j$ defaults
- Equity as capital buffer $E_i$:
  Bank defaults when $E_i \leq \gamma (= 0)$
Default contagion

Contagion dynamics:

- Bank $i$ faces loss $A_{ij}$ if $j$ defaults
- Bank $i$ defaults when $A_{ij} > E_i (-\gamma)$
- Define impact of $j$ on $i$:

$$W_{ji} = \min\{1, \frac{A_{ij}}{E_i}\}$$

Note: Impact runs in opposite direction

- Distress: Impact $< 1$
- Outright default: Impact $= 1$

- Economic value of bank $i$: $v_i = \frac{A_i}{\sum_k A_k}$
DebtRank

- Dynamic variable attached to each node $i$:
  - Continuous impact $h_i \in [0, 1]$  
  - Discrete state $s_i \in \{U, D, I\}$
    - Undistressed, distressed, inactive

- Initial conditions when nodes $i \in S$ are distressed:

\[
\begin{align*}
h_i(1) &= \begin{cases} 
\psi & i \in S \\
0 & i \notin S 
\end{cases} \\
s_i(1) &= \begin{cases} 
D & \text{if } i \in S \\
U & \text{if } i \notin S 
\end{cases}
\end{align*}
\]

Initial distress $\psi \in (0, 1]$, $\psi = 1$ denotes default
DebtRank

- Dynamics of distress spreading:

\[
    h_i(t) = \min \left\{ 1, h_i(t - 1) + \sum_{j: s_j(t - 1) = D} W_{ji} h_j(t - 1) \right\}
\]

\[
    s_i(t) = \begin{cases} 
    I & \text{if } s_i(t - 1) = D \\
    D & \text{if } h_i(t) > 0 \land s_i(t - 1) \neq I \\
    s_i(t - 1) & \text{otherwise}
    \end{cases}
\]
DebtRank

- Dynamics of distress spreading:

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s_i(t-1) & \text{otherwise}
\end{cases} \]

- Spreading stops after finite time \( T \) with all nodes in state \( U \) or \( I \). Total impact accumulated in \( h_i(T) \).

- **DebtRank** defined as

\[ R = \sum_i h_i(T)v_i - \sum_i h_i(1)v_i \]

Measures distress *induced* in the system, excluding initial distress.
N <- length(V(G))
v <- rep(1, N)/N
debt.rank(G, v, c(1), plot = TRUE)
DebtRank

Data used in paper

- Top 22 financial institutions borrowing from FED during crisis.
- TNC network of equity holdings $Z$ as proxy for lending:
  - Dense network: 58% of all links
  - Avg. core number $\approx 8$

\[
\tilde{W}_{ij} = \frac{Z_{ji}}{\sum_l Z_{jl}}
\]

\[
W_{ij} = \max \left\{ 1, \alpha \frac{\tilde{W}_{ij}}{\max_l \tilde{W}_{lj}} \frac{E_j(1)}{E_j(t)} \right\}
\]

Banks can withstand default of at least $\frac{1}{\alpha}$ counterparties.
Here, $\alpha = \frac{1}{5}$. 
DebtRank increased during crisis:

- **a)** February 2008
- **b)** November 2008
DebtRank correlates weakly with bank size:

Fragility denotes leverage, i.e. $\frac{\text{Debt}}{\text{MarketCap}}$
DebtRank makes several assumptions

- Distress spreading stops after first hit to bank
- Distress spreads before banks default

Other assumptions are plausible/possible as well...
Financial contagion

DebtRank makes several assumptions

- Distress spreading stops after first hit to bank
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Other assumptions are plausible/possible as well . . .

Model by Eisenberg & Noe (2001):

- Liabilities honored in full if bank is solvent
- Default only spreads if banks default
Financial contagion

Model by Eisenberg & Noe (2001):

- Net asset values of bank $i$

$$A_i^E + A_i^I - L_i^E$$

Note: External liabilities senior to interbank liabilities

- Repayment vector $I$ solves

$$I_i = \min\{A_i^E - L_i^E + \sum_j \Pi_{ji} I_j, L_i^I\}$$

with

- nominal interbank liabilities $L_i^I = \sum_j L_{ij}^I$
- relative debt holding fractions $\Pi_{ji} = \frac{L_{ij}^I}{L_j^I} = \frac{A_{ij}^I}{L_j^I}$
Network valuation

Simple example . . . initial state

Figure 3. A Hypothetical Financial Network Showing Payments Due

Note: The net worth of each node is shown in bold. The outside sector consists of households and nonfinancial firms.

- All banks solvent
- Liabilities honored in full

Fig. from: P. Glasserman & H. Peyton Young, Contagion in Financial Networks, JEL 2016.
Network valuation

Simple example . . . after shock to asset of bank C

- All banks in default
- Liabilities repaid pro rata

Fig. from: P. Glasserman & H. Peyton Young, *Contagion in Financial Networks*, JEL 2016.
Network valuation

Most models of default contagion can be considered as *network valuations*

\[ E_i = A_i^E + \sum_j A_{ij}^I \vee (E_j) - L_i^E - \sum_j L_{ij}^I \]

- Liabilities valued at face value
- Asset values
  - at market prices (external)
  - adjusted to solvency of counterparty (internal)
Network valuation

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E_i = A_i^E + \sum_j A_{ij}^I \vee (E_j) - L_i^E - \sum_j L_{ij}^I
\]

- Liabilities valued at face value
- Asset values
  - at market prices (external)
  - adjusted to solvency of counterparty (internal)
- Firm valuation
  - Market equity \( \max\{E_i, 0\} \)
  - Market value of debt \( \min\{L_i^I, E_i + L_i^I\} \)
  - Market value of net assets \( E_i + L_i^I \)
Network valuation

Most models of default contagion can be considered as network valuations

\[ E_i = A_i^E + \sum_j A_{ij}^I \mathbb{V}(E_j) - L_i^E - \sum_j L_{ij}^I \]

- Liabilities valued at face value
- Asset values
  - at market prices (external)
  - adjusted to solvency of counterparty (internal)

Leads to self-consistent equity valuation

\[ \mathbf{E} = \Phi(\mathbf{E}) = \mathbf{A}^E + \mathbf{A}^I \mathbb{V}(\mathbf{E}) - \mathbf{L}^E - \mathbf{L}^I \]
Network valuation

Model by Eisenberg and Noe (2001)

► Repayment vector \( l \) solves

\[
l_i = \min\{A_i^E - L_i^E + \sum_j \frac{A_{ij}^l}{L_j} l_j, L_i^l\}
\]

Note: This is the market value of debt

► We can rewrite this as

\[
E_i + L_i^l = A_i^E - L_i^E + \sum_j \frac{A_{ij}^l}{L_j} l_j
\]

Note: \( E_i < 0 \iff l_i < L_i^l \)

► Thus, we obtain valuation

\[
E_i = A_i^E + \sum_j A_{ij}^l \mathbb{V}(E_j) - L_i^E - L_i^l
\]

where \( \mathbb{V}(E_j) = 1_{E_j \geq 0} + 1_{E_j < 0} \left( \frac{E_j + L_j^l}{L_j^l} \right)^+ \)
Network valuation

- Model by Eisenberg and Noe (2001)

\[ \nabla(E_j) = \mathbb{1}_{E_j \geq 0} + \mathbb{1}_{E_j < 0} \left( \frac{E_j + L_j'}{L_j'} \right)^+ \n\]

- Ex-ante version of this model, i.e. \( E_i(t) = \mathbb{E}^Q[E_i(T)|\mathcal{F}(t)] \)

with valuation

\[ \nabla(E_j) = \mathbb{E}[\mathbb{1}_{E_j \geq 0}|A^E(t)] + \mathbb{E} \left[ \mathbb{1}_{E_j < 0} \left( \frac{E_j + L_j'}{L_j'} \right)^+ \right| A^E(t) \]

Under Black-Scholes model we could evaluate risk-neutral probabilities of solvency and expected recovery.

- Linear DebtRank model

\[ \nabla(E_j) = \frac{E_j}{E_j(0)} \]

- Propagates distress \( h_j(t) = \frac{E_j(0) - E_j(t)}{E_j(0)} \)

- Distress can hit bank multiple times
Network valuation

Self-consistent valuation

\[ E = \Phi(E) \]

Models differ in valuation functions

![Diagram showing interbank valuation functions as a function of the equity of the borrower.](image)

Figure 1: Interbank valuation functions as a function of the equity of the borrower. Parameters as follows. EN: \( \bar{p} = 2 \), Furfine: \( R = 1 \), Linear DebtRank: \( M = 2.5 \), Ex-ante EN: \( A^e = 1, \bar{p} = 2, \beta = 1, \sigma = 1 \).

Commonly, numerical studies illustrate contagion effects

Network valuation

Self-consistent valuation

\[ E = \Phi(E) \]

Mathematical questions

▶ When does fixed point exist?
  Standard contracts lead to monotone valuation
  \[ \implies \text{fixed-point exists by Knaster-Tarski theorem} \]

▶ Is the fixed-point unique?
  Several sufficient conditions given in the literature . . .
  Economic arguments in favor of greatest fixed-point

▶ Which bank is most systemically relevant (central)?
  \textit{DebtRank}: Numerical impact of each bank if shocked initially
Risk management

Equity values can be considered as derivative contracts\(^1\)

\[ E(A^E) = \Phi(E; A^E) \]

on value of external assets \( A^E \).

- Recall Merton’s structural credit risk model

\[ E = \max\{A - L, 0\} \]

Equity is implicit call option on firm’s asset values \( A \) with strike \( K = L \) (under certain assumptions)

- Network valuation generalizes this insight to multiple firms with interbank liabilities

\(^1\)Such as options, futures, \ldots
Risk management

Equity values can be considered as derivative contracts\(^1\)

\[
E(A^E) = \Phi(E; A^E)
\]

on value of external assets \(A^E\).

**Greeks:** Vital tool in risk management

- Partial derivatives of contract value
  - Spot price \(A^E\)
  - Volatility \(\sigma\)
  - Value \(E\)
  - Delta \(\Delta\)
  - Vega \(\nu\)
  - Gamma \(\Gamma\)
  - Vanna
  - Vomma

- Quantify sensitivities to different risks

\(^1\)Such as options, futures, ...
Risk management

What about network $\Delta$ . . .

$$E = \Phi(A^E, E) = A^E + A^I \nabla(E) - L^E - L^I$$

Taking derivatives on both sides

$$\frac{\partial E_i}{\partial A^E_k} = \frac{\partial}{\partial A^E_k} \left( A^E_i + \sum_j A^I_{ij} \nabla(E_j) - L^E_i - L^I_i \right)$$

$$= 1_{i=k} + \sum_j A^I_{ij} \nabla'(E_j) \frac{\partial E_j}{\partial A^E_k}$$

and switching back to matrix notation

$$\frac{\partial E}{\partial A^E} = I + A^I \nabla'(E) \frac{\partial E}{\partial A^E}$$

$$= \left( I - A^I \nabla'(E) \right)^{-1}$$
Risk management

What about network $\Delta$ ... 

$$E = \Phi(A^E, E) = A^E + A^I \nabla(E) - L^E - L^I$$

- Network amplifies initial asset price shock $\frac{\partial A^E}{\partial A^E} = I$ via

$$\left(I - A^I \nabla'(E)\right)^{-1}$$

- Modified Katz centrality where each column $A^E_j$ of cross-holding matrix is weighted by $\nabla'(E_j)$. Note: $\nabla'(E) = \frac{\partial \nabla(E)}{\partial E}$ is $\Delta$ of valuation adjustment.
Figure: Greeks $\Delta$ and $\mathcal{V}$ for Erdős-Rényi random cross-holding networks depending on the spot price $a_0$ of external assets. In all cases the average impact of each firm on the equity, debt and value of all other firms is shown.

DebtRank-transparency: Controlling systemic risk in financial networks

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Nodes in a financial network, such as banks, cannot assess the true risks associated with lending to other nodes in the network, unless they have full information on the risks of all other nodes. These risks can be estimated by using network metrics (as DebtRank) of the interbank liability network. With a simple agent based model we show that systemic risk in financial networks can be drastically reduced by increasing transparency, i.e., making the DebtRank of individual banks visible to others, and by imposing a rule, that reduces interbank borrowing from systemically risky nodes. This scheme does not reduce the efficiency of the financial network, but fosters a more homogeneous risk distribution within the system in a self-organized critical way. The reduction of systemic risk is due to a massive reduction of cascading failures in the transparent system. A regulation-policy implementation of the proposed scheme is discussed.

Since the beginning of banking the possibility of a lender to assess the riskiness of a potential borrower has been essential. In a rational world, the result of this assessment determines the terms of a lender-borrower relationship (risk-premium), including the possibility that no deal would be established in case the borrower appears to be too risky. When a potential borrower is a node in a lending-borrowing network, the node's riskiness (or creditworthiness) not only depends on its financial conditions, but also on those who have lending-borrowing relations with that node. The riskiness of these neighboring nodes depends on the conditions of their neighbors, and so on. In this way the concept of risk loses its local character between a borrower and a lender, and may become systemic. A systemic risk node in a financial network is one that should it default – will have a substantial impact (losses due to failed credits) on other nodes in the network. Note that this notion of systemic risk is different from the risk of not getting paid back, the credit default risk.

The assessment of the systemic riskiness of a node turns into an assessment of the entire financial network. Such an exercise can only be carried out with information on the asset-liability network. This information is, up to now, not available to individual nodes in that network. In this sense, financial networks – the interbank (IB) market in particular – are opaque. This in transparency makes it impossible for individual banks to make rational decisions on lending terms in a financial network, which leads to a fundamental principal opacity in financial networks rules out the possibility of rational risk assessment, and consequently, transparency, i.e. access to system-wide information is a necessary condition for any systemic risk management. Note that recently an alternative notion for systemic importance of banks for the fluid transmission of credit through the IB market has been discussed in terms of “controllability”.

The banking network is a fundamental building block in our globalized society. It provides a substantial part of the funding and liquidity for the real economy. The real economy – the ongoing process of invent, production, distribution, use, and disposal of goods and services – is inherently risky and introduces a third type of risk, the economic risk. This risk originates in the uncertainty of payoffs from investments in business ideas, which might not be profitable, or simply fail. This economic risk can not be eliminated from an evolving economic system, however it can be spread, shared, and diversified. One of the roles of the financial systems is to distribute the risk generated by the real economy in the economy among the actors in the financial network. The financial network can be seen as a service to share the burden of economic risk. By no means should this service by itself produce additional systemic risk on top of economic risk endogenously. Neither should the design and regulation of financial networks introduce mechanisms that leverage or induce the economic risk. As long as systemic risk is endogenously generated within the financial network, this system is not yet properly designed and regulated. In this paper we show that, unless a certain level of transparency is introduced in financial networks, systemic risk will be endogenously generated within financial networks. Systemic risk is hard to reduce with traditional regulation schemes5. By introducing a minimum level of transparency in financial networks, endogenous risk can be drastically reduced without negative effects on the efficiency or volume in the financial services for the real
DebtRank Transparency

Bank issue loans to firms:

- **Normal mode**: Random bank issues loan
- **Transparent mode**: Loan from bank with lowest DebtRank
- **Fast mode**: As above, but DebtRank recomputed after each transaction
DebtRank Transparency

Transparency stabilizes banking system:

Note: Katz centrality works just as well