Solution 1. The cluster coefficient can be interpreted in statistical terms as

\[ C = \mathbb{P} \{ j - k \mid i - j, i - k \} \]

In the configuration model edges are independent and any pair of vertices with degrees \( k_i \) and \( k_j \) is connected with probability \( \frac{k_i k_j}{2m} \). To compute the clustering coefficient, we need to take into account that both \( j \) and \( k \) are already reached by an edge. Thus,

\[
C = \frac{1}{2m} \left( \sum_k q_k k \right)^2 = \frac{1}{n \langle k \rangle} \left( \sum_k \frac{(k+1)p_{k+1}}{\langle k \rangle} k \right)^2 = \frac{1}{n} \frac{(\langle k^2 \rangle - \langle k \rangle)^2}{\langle k \rangle^3}
\]

where we used that \( 2m = n \langle k \rangle \) and \( q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle} \).

Solution 2. Consider the global cluster coefficient. We need to count triangles and two-stars.

The number of two-stars, containing vertex \( v_i \) as middle vertex, is given as \( \frac{k_i(k_i-1)}{2} \) where \( k_i \) denotes the degree of \( v_i \). Thus, the total number of two-stars is \( n_\Delta = \frac{1}{2} \sum_i k_i(k_i-1) \).

Let \( n_{\Delta,i} \) denote the number of triangles containing vertex \( v_i \). Then, the total number of triangles is \( n_\Delta = \frac{1}{3} \sum_i n_{\Delta,i} \), since by symmetry each triangle gets counted 3 times.

Thus,

\[
C = \frac{3n_\Delta}{n_\Delta} = \frac{3}{2} \sum_i \frac{n_{\Delta,i}}{k_i(k_i-1)} = \sum_i \frac{1}{2} \frac{k_i(k_i-1)}{n_\Delta} C_i
\]

The global clustering coefficient corresponds to a weighted average of the local clustering coefficient, where each vertex contributes according to its two-star density \( \frac{k_i(k_i-1)}{n_\Delta} \).

The following example illustrates this difference between the local and global clustering coefficient:
tri.star <- function (n) {
  (make_ring(3) * n) %>%
    add.vertices(1) %>%
    add.edges(c(sapply(3*seq(1,n), function(x) c(3*n+1,x))))
}

plot(tri.star(5),
     vertex.label = NA,
     vertex.color = 'lightblue',
     vertex.size = 10)
**Solution 3.** 1. The SBM represents a graph with two groups/types of vertices which exhibit disassortative mixing.
2. To compute the average degree, we first construct the graphon \( W \) corresponding to this SBM:

\[
W(u, v) = \begin{cases} 
\frac{1}{5} & \text{for } u \leq \frac{1}{5}, \text{ and } v \leq \frac{1}{5}, \\
\frac{3}{25} & \text{for } u \leq \frac{1}{5}, \text{ and } v > \frac{1}{5}, \\
\frac{4}{25} & \text{for } u > \frac{1}{5}, \text{ and } v \leq \frac{1}{5}, \\
\frac{1}{4} & \text{for } u > \frac{1}{5}, \text{ and } v > \frac{1}{5}.
\end{cases}
\]

The degree distribution is represented by

\[
f(U) = \int_0^1 W(U, v) \, dv,
\]

i.e. the fraction of connections to vertex at latent position \( U \).

The average degree in a network of size \( N \) is then given as

\[
\langle k \rangle = \mathbb{E}[Nf(U)] = N\mathbb{E}\left[\int_0^1 W(U, v) \, dv\right]
\]

\[
= N \left( \frac{1^2}{5} + \frac{143}{5 \cdot 54} + \frac{4^2}{5 \cdot 4} \right)
\]

\[
= N \left( \frac{4 + 72 + 48}{25 \cdot 12} \right) = N \frac{31}{75}.
\]

3. To compute the clustering coefficient we need the subgraph counts for connected triplets and triangles, i.e. \( t(\vee, G) \) and \( t(\Delta, G) \).

Note that since we count homomorphisms the clustering coefficient is defined as \( \frac{t(\Delta, G)}{t(\vee, G)} \).

The homomorphism counts are obtained as follows:

\[
t(\vee, G) = \int_0^1 \int_0^1 \int_0^1 W(u_1, u_2)W(u_2, u_3) \, du_1 du_2 du_3
\]

\[
= \frac{1}{5} \left( \frac{1}{5} \left( \frac{1}{5} \left( \frac{1}{5} \left( \frac{1}{5} \cdot 3 \cdot \frac{3}{4} \cdot \frac{3}{4} \right) + \frac{4}{5} \left( \frac{1}{5} \cdot \frac{3}{4} + \frac{4}{5} \cdot \frac{1}{4} \right) \right) \right) \right)
\]

\[
+ \frac{4}{5} \left( \frac{1}{5} \left( \frac{13}{5} \cdot 1 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} + \frac{4}{5} \left( \frac{11}{5} \cdot 3 \cdot \frac{4}{5} \cdot \frac{1}{4} \right) \right) \right)
\]

\[
= \frac{1}{125} \left( \frac{1}{5} + \frac{9}{4} + \frac{3}{4} + 1 + 9 + 3 + 4 \right)
\]
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N. Bertschinger

6. Solution

and

\[ t(\Delta, G) = \int_0^1 \int_0^1 \int_0^1 W(u_1, u_2)W(u_2, u_3)W(u_1, u_3) \, du_1 du_2 du_3 \]

\[ = \frac{1}{5} \left( \frac{1}{5} \left( \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \right) + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} \right) + \frac{4}{5} \left( \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \right) \]

\[ + \frac{4}{5} \left( \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \right) \]

\[ = \frac{1}{125} \left( \frac{1}{27} + \frac{3}{4} + \frac{9}{4} + \frac{3}{4} + \frac{9}{4} + \frac{9}{4} + \frac{1}{4} \right). \]

Combining these counts we obtain

\[ \frac{t(\Delta, G)}{t(\lor, G)} = \frac{1/108}{1/36} \left( 4 + 81 + 81 + 243 + 81 + 243 + 243 + 108 \right) \]

\[ \approx \frac{271}{570} \approx 0.475 \]

Solution 4. The probability \( \mathbb{P}(G_n) \) of graph \( G_n \) with adjacency matrix is given as

\[ \mathbb{P}(G_n) = \int_0^1 \cdots \int_0^1 \prod_{ij} \text{Bernoulli}(A_{ij}|W(u_i, u_j)) \, du_1 \cdots du_n. \]

Writing \( \beta_i = g(u_i) \) the integrand can be simplified to

\[ \prod_{ij} \text{Bernoulli}(A_{ij}|W(u_i, u_j)) = \prod_{ij} W(u_i, u_j)^{A_{ij}} (1 - W(u_i, u_j))^{1 - A_{ij}} \]

\[ = \prod_{ij} \left( \frac{\beta_i \beta_j}{1 + \beta_i \beta_j} \right)^{A_{ij}} \left( \frac{1}{1 + \beta_i \beta_j} \right)^{1 - A_{ij}} \]

\[ = \prod_{ij} \left( \frac{\beta_i \beta_j}{1 + \beta_i \beta_j} \right)^{A_{ij}} \left( \frac{1}{1 + \beta_i \beta_j} \right)^{1 - A_{ij}} \]

\[ = \prod_{i=1}^n \prod_{j=1}^n \frac{(\beta_i \beta_j)^{A_{ij}}}{1 + \beta_i \beta_j} \]

\[ = \frac{\prod_{i=1}^n \beta_i^{k_i}}{\prod_{j=1}^n (1 + \beta_i \beta_j)} \]

showing that it only depends on the vertex degrees. As this also holds after integrating out the latent variables, we obtain the desired result.
Note: The $\beta$-model assigns a latent affinity value $\beta_i$ to each vertex. Edges are then drawn independently with probability

$$p_{ij} = \frac{\beta_i \beta_j}{1 + \beta_i \beta_j}$$

which is equivalent to the above graphon model, if $\beta$ is distributed as $\beta \overset{d}{=} g(U)$. 