Exercise 1. Configuration model

Consider the degree sequence $(3, 2, 1)$. Give all (unlabeled) graphs and their probabilities under the configuration model.

Hint: Remember that the configuration model can produce self-loops and multi-edges.

Exercise 2. Giant component

- Give the formula for the size of the giant component in the Erdős-Rényi model.
- For the configuration model, we showed that the size of the giant component $S$ solves

$$S = 1 - g_0(u) \quad \text{and} \quad u = g_1(u)$$

Show that this reduces to the formula for the Erdős-Rényi model when assuming a Poisson degree distribution.

Hint: The generating function of the Poisson distribution is given by

$$g_0(z) = e^{\lambda (z-1)}$$

where $\lambda > 0$ equals the mean degree $\langle k \rangle$.

Exercise 3. Erdős-Rényi model

- The fraction $S$ of vertices in the giant component is given by the (non-zero) solution of

$$S = 1 - e^{-\langle k \rangle S}$$

Solve this equation numerically and plot $S$ as a function of the average degree $\langle k \rangle$:

- What happens at $\langle k \rangle = 1$?
- Why is it called a phase transition?

- Compare the diameter of Erdős-Rényi random graphs with lattice graphs of dimension 1, 2 and 3:

- What do you observe?
Why are random graphs said to be “small world”?

Note: Lattice graphs are regular graphs whose vertices correspond to points with integer coordinates in $\mathbb{R}^d$. Neighbouring points, i.e. of Euclidean distance one, are connected by an edge.

Exercise 4. Configuration model

Implement the configuration model in R. That is, write a function with input

- $(k_i)_{i=0}^{n}$ vector of vertex degrees

and output

- a graph (in igraph format) randomly drawn according to the configuration model

Note: Your function should throw an error if the given degree sequence is not graphical, i.e. no valid graph exists.